Mathematical Games:

A Criticism of Chess, Checkers and Bridge

By ARTHUR BERNHART

In *The Theory of Games and Economic Behavior,* Von Neumann and Morgenstern introduce a static theory of games. The approach is so abstract, however, that although chess is mentioned a dozen times, all their comments are equally true of tic-tac-toe. But generalization is only one aspect of mathematics, and any analysis of chess is incomplete which does not consider those characteristics which distinguish it from other similar games, such as checkers and bridge. In this brief survey of three typical games of skill, the insight and the limitations of a dynamic mathematical theory are considered.

Should mathematicians make good chess players? It would seem so, but paradoxically they seldom play chess. Conversely, chess players are apt to be doctors, lawyers, merchants, perhaps teachers of music or literature, but seldom of mathematics or science. Why is this? "Chess is a busman's holiday," says Professor N.-A. Court, who abandoned the game for research in modern geometry. The mathematics teacher finds an intellectual tug-of-war in professional research, and has no need for chess or bridge. Plato said, "I do not live to play, but play in order that I may live, and return with greater zest to the labors of life."

The process of making a good chess move combines the creative imagination of geometric intuition with the logical technique of algebraic manipulation. However, the chess player must be content with *partial* solutions, for even with the help of an electronic robot, it would take more than a lifetime to analyze every possibility. After each side has moved only twice, already 67,259 different situations have arisen. In a typical mid-game situation there is even greater multiplicity. The chess player scans the board with his imagination alerted for artistic combinations. From the welter of possible continuations he somehow selects a very few for further study. Then with scientific precision he probes the chosen few for likely lines of play and for hidden intricacies. On the basis of his findings he chooses a move. Even this final choice is highly personal. Thus, for example, suppose from the current situation A, a player has three moves, B_1, B_2, and B_3. Suppose further that his opponent must respond to the first B_1 with C_1, may reply to the second B_2 with either C_1 or C_2, and could meet the third B_3 with C_3 or C_4. If the first contingency, C_1, were favorable, he could choose B_1 and thereby compel a favorable outcome. But if C_1 is mediocre and C_2 favorable, with C_3 and C_4 of unknown quality, he may choose B_2. If he deems C_2 actually more favorable than C_1, where a cursory examination might decide otherwise, his strategy may be to stump his opponent with the problem B_2, tempting him to blunder; for "the game is won by the player who makes the next to the last mistake." A contrary strategy assumes your opponent will find his best reply, and urges you to "beat his best moves not his mistakes." Some timid players would therefore choose B_1 and certain mediocrity, avoiding de Maupassant's black "Door"; where others of more pioneering spirit would risk B_3 and the unknown vicissitudes of C_3 and C_4. In our simplified example with a choice at A among three moves, B, and even with the selfsame information (or lack of information) on the further consequences, C, three good players might reach three different decisions.

If chess is complicated, chess personality is more complex. It occasionally happens that among three players, each wins from one but loses to the other, as in the "rock breaks scissors, scissors cuts paper, paper covers rock" triangle. In the Norman Chess Club of some dozen players, one man prefers a beautiful loss to a trite win, declaring that "it is unethical to seek a mathematically sure win." A second player will never sacrifice tangible force for intangible positional superiority, for "he who does not hesitate is lost"; while a third player habitually attacks violently from strategically weak formations, believing "the best defense is a good offense." When one member of the club, who plays the opening with meticulous care, is matched with another noted for stubborn endings, we have mid-game combinations of kaleidoscopic beauty.

Chess is a game for two players. An investment of ten dollars in a good chess set, each piece weighted and felted, will provide a lifetime of enjoyment; though one can learn with a fifty-cent expenditure—or trade in the family Buick for rare and beautiful pieces. An average game involves from 40 to 60 moves on each side, lasting one or two hours, but a crucial match or tourna-

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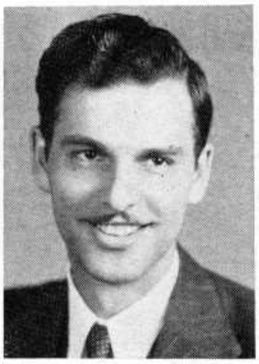

Dr. Bernhart, Associate Professor of Mathematics, knows what he is talking about when he discusses, chess, checkers, and bridge. In chess, for instance, he played on the Norman team which placed second and first in two inter-city matches (Tulsa, Oklahoma City, Stillwater, Bartlesville, and Norman); and he has a tie to his credit in an exhibition game with Koltanowski, world professional chess player. In checkers, he has recorded ties in tourneys with champions from Kansas, Missouri, and Oklahoma. He has also tied in exhibition games with the well-known professionals, Newell Banks, of Detroit, and Willie Ryan, of New York City.

Bridge he plays for fun.

ment game may take twice as long. When faced with certain defeat the loser usually resigns, but if the game is played out to its conclusion, the winner has the satisfaction of announcing "checkmate," derived from the Persian *shah mat,* meaning "the king is dead." A fool can be mated in two moves, and many a tyro has submitted to "Scholar's Mate" at the fourth move.

Perhaps the quickest mature win against a worthy opponent begins with four natural developing moves, sacrifices a queen on the next move, and (if the queen gambit is accepted) forces checkmate at the seventh move. "Beware of Greeks bearing gifts." Whenever the enemy king is put in immediate danger of capture, it is chivalrous to give a warning "check." Since it does not profit a man to gain the whole board and lose his own king, the "check" places a restraint on his aggressive plans, putting him on the defensive. By analogy, casting out nines is a "check" for addition, a horse is curbed with a checkrein, and Shylock kept a record of the bills of exchange he issued in lieu of gold. In more recent times, the name of the Persian king has passed to the bill of exchange, the record is renamed "check stub," and the gold is buried at Fort Knox. The word "chess" is from the French plural of check.

The game is a mimic warfare, except that both sides have equal strength, and each contestant is fully aware of the disposition of forces on the field of battle. It is a game of pure skill, with no element of luck to upset the rationally predictable consequences. In spite of this equality of opportunity, there are very few tie results except in big tournaments where masters play, when a third or sometimes even half the games are drawn. In the Alekhine-Capablanca match of 1927 the score was 6 to 3 with 25 draws. If the forces are decimated so that one player is left with a lone king, the other king may win with the help of either a queen or rook. But a single knight or bishop cannot force mate, and for this reason knights and bishops are called minor pieces. With minor pieces and pawns it takes two to give mate, though if the lone king is out of position, a single pawn may suffice, but then only by reaching the eighth rank and being promoted to a major piece. Since either player may challenge the other to win by checkmate within fifty moves (no pawn moved or piece taken on either side), and since stalemate is scored as a tie result, a well-placed king can tie against a single pawn or even against two knights. The combination of a knight and bishop is just sufficient to force checkmate, but then only by driving the defending king into one of the corners of the board controlled by the bishop. Almost the full quota of fifty moves is required for this beautiful win, and one clumsy maneuver may permit the lone king to escape. By contrast, the wins with two bishops, and with a major piece, are relatively simple.

In estimating the advisability of various mutual captures, a customary yardstick counts a queen worth ten pawns, or two rooks, or three minor pieces. This rule obviously values a rook at five pawns, but less than a minor piece and two pawns. Similarly, either minor piece rates better than three pawns, with some players giving a preferential nod to the bishop. If these conversion tables were valid for all positions, chess would lose much of its appeal. Positional plus or minus is difficult to measure in a satisfactory way. Masters often disagree as to when there is equality of position. Granted that one side has an advantage, can we measure this in pawn units? If the other side can escape from his disadvantage by giving up two pawns but not with the sacrifice of only one, it seems natural to evaluate the difference of position at nearly two pawn units. But he may not wish to buy equality at that price. Perhaps I can liquidate my advantage for one pawn unit. If I am willing to do so, my advantage is nearly one pawn unit. He can escape for two pawn units with or without my permission, and I can ignore his wishes and settle for one pawn unit. If these upper and lower measures agreed, we would have a standard measure of position, but where there is a significant discrepancy our arithmetic does not apply. Since a rook advantage is more than enough to force checkmate, it is plausible to set four pawn units as the measure of any winning position. Some chess theorists count white's advantage at having the first move as one-third of a pawn unit, and I have known good chess players who methodically hoard these units. But enough has been said here to show that too much attention to tactics is poor strategy, lest with Pyrrhus you win battles and lose the war. In chess as in mathematics, do not sell your artistic birthright for a mess of computational pottage.

Much that has already been said concerning chess could be repeated verbatim for checkers. Like two great universities in the same state, their similarity has led to a traditional rivalry. Which is the better game? Each is too deep for a complete crossboard analysis, so the claim that chess is the deeper is meaningless. Victor Hugo compares the checkerboard with the Cathedral of Notre Dame, whose architecture is distinguished by "that something surpassingly grand in the simple and striking in the beautiful." Edgar Allan Poe in "Murders in the Rue Morgue" praises checkers as more profound, libelling the greater complexity of chess as an "elaborate frivolity" of "bizarre motions." In reprisal, one chess club awarded a book titled "How to Play Checkers" as booby prize in its annual tournament. As for me, I am fond of both games. Which is the better pie, apple or pumpkin?

Checker sets are more economical: three dollars will purchase beautiful red and white plastic interlocking pieces, with a buff and green checkerboard. (Do not buy red and black checkers for a red and black board!) The laws of play are much simpler. It takes less time to play a game, usually less than an hour. But these are superficial differences which the mathematical theory ignores. Let us consider five points of intrinsic significance.

(1) There is a greater possibility for a checker game to end in a tie. For instance, when Newell Banks of Detroit played a fifty game match with Robert Stewart of Scotland for the championship of the world, the final score was Stewart 2 wins, Banks 1 win, and 47 draws.

(2) There are longer chains of deductive reasoning. In a standard ending known as "First Position" both sides have a king and a man, but the side with the positional advantage can force a win after 60 moves. The checker player is often called upon to choose between two moves that seem alike, but where one wins and the other loses, as becomes evident thirty moves later. In this sense checkers is the more difficult game.

(3) Normally an extra checker gives one an overwhelming advantage. Thus, twelve men against eleven is a sure win. The side with the extra man merely trades down. Eventually 4 vs. 3 becomes 3 vs. 2, then 2 vs. 1, and even the double corner is no refuge. A checker is often temporarily sacrificed for a quick king, but unless the man is soon regained, the game is lost. Chess, on the other hand, owes most of its charm to the sacrificial gambit. Another way of putting the matter is this: a chess game is so long that a player has time to recoup his losses in position or material; a checker game is too short for intangible investments, for one mistake is usually fatal.

(4) No treatment of checkers is adequate without mentioning the subtlety of the odd-move. Its mathematical basis is the fact that a checker always takes an even number of moves in any circuit which returns to its starting point. Consider any square. The sixteen squares which can be reached in an odd number of moves form one system; the other sixteen squares which can be reached in an even number of moves form another system. When it is your turn to move, count the number of occupied squares belonging to either system. If that number is odd, you are said to have the odd-move, otherwise you have the even-move. In any maneuvering for position which

An Analysis of Bridge Provides Interesting Paradoxes

does not involve jumping, the same player will retain the odd-move. Some jumps change the odd-move, others do not. A scientific player knows which traps work when you have the odd-move, and which work with the even-move, and plans his play accordingly.

(5) A final comment will suffice for this brief criticism of checkers. In the theory of any game there is a conflict between particular tactics and general strategy. As an aid to remembering the many details, theorists formulate *rules of play* which are intended to guide the player in making his choice of moves. For example, the chess player is urged to develop his pieces before launching an attack, to double his rooks on open files, to avoid doubling his pawns, and so on. The checker player is urged to move to the center and not to the side, to defend his double corner, or to keep two men in his king row at the first and third squares. One should remember that the peculiarities of a particular situation take precedence over such general rules. "The exception *probes* the rule." The ability to sense when the rule of thumb does not apply, and to use these exceptions to undermine the plans of the enemy, is the trademark of the critical player.

As we turn our attention to bridge we are confronted with two additional factors: (1) partnership co-operation, and (2) the element of chance. In the original game of whist, four players struggled to take tricks, those taken by either partner counting for the partnership. As there are thirteen tricks in each deal, each player will average three (plus one-fourth), and any tricks a partnership takes beyond the first six, is above the average. Points are scored for each extra trick, the object of the game being to make the most points.

Bridge introduced the innovation of the dealer playing his partner's cards, which are turned face up for all to see. This *bridge* between dealer and dummy guarantees one hundred per cent co-operation, but the opposing players can only guess what each other holds and their team play is less effective. The trump suit had been determined by lot, but in Auction the privilege of naming trump and playing dummy was given to the highest bidder. A bid of "two hearts" is a promise to take two extra tricks (above the "book" of six) if hearts are trump. The suits were ranked in alphabetical order: clubs (lowest), diamonds, hearts, spades, no trump (highest) so that "two spades" became a higher bid than "two hearts," and "three clubs" next higher than "two no trump." The points awarded for tricks above book were staggered from six points with clubs trumps to ten points a trick at no trump. An extra "game" bonus was awarded for earning thirty points, or else a "rubber" bonus for the side first to win two games.

These points are forfeited if the side fails to make as many tricks as bid, and a penalty score is added to the opponents' score. Additional points are awarded for taking all thirteen tricks (grand slam), or all but one (small slam). Point credit for holding certain high card combinations (honors) in the trump suit is given to the original holder regardless of the play. Finally, the highest bidder gets to play last on the first trick, instead of first. A popular recent variation of bridge is Contract. All auction scores are roughly multiplied by three. In addition, game and slam bonuses are awarded only if they are *contracted* for in the bid. Auction had already contained a device (doubling and redoubling) for increasing the point difference between success and failure, and Contract widened the breach. Every few years the scoring is modified, and a new series of explanatory books hits the market.

The historical perspective enables one the better to appreciate this excellent card game. This study is based on the assumption that the object of the game is to win points. This is stated at the outset not to deny that some groups de-emphasize the score and emphasize social conversation, refreshments, or prizes, but to limit the number of intangibles so that some sort of mathematical analysis can be made. After all, any game can be played with secondary objectives, but the conditions for winning give a game its orientation and distinguish it from other games. Low score wins in golf, with a secondary objective of getting a walk in the fresh air. Should the secondary objective become primary, one is walking and not golfing. Many children play checkers with supplementary goals: beyond winning they try to win with the largest number of pieces, or after a loss is certain they continue to play to see if they can get a king. But in the adult game all wins are scored equally, whether due to overwhelming superiority or to a fluke in the ending. Professional athletic teams sometimes try to hold the score down to increase gate receipts, knowing the public prefers a closely contested match, but here we shall consider bridge strictly from an amateur point of view.

The paramount importance of points provides some paradoxes as interesting as a presidential candidate losing the election with a popular majority of votes. It is possible to win two out of three games in the rubber and lose on total points, as Hoyle is careful to mention. With a sufficient margin in the score of an unfinished rubber, it is advantageous to deliberately throw the last game in order to win. Does this remind you of a baseball team trying to strike out before rain or curfew? Culbertson, in commenting on the large penalties for overbidding, particularly when set by more than one trick, vulnerable and doubled, goes so far as to say that the object of the game is not bidding in order to win the contract, but, by threatening to do so, tempting the opposing side to overbid! Since the bridge laws provide different scoring opportunities depending on how many points are required for game and how many games are needed for rubber, it follows that different strategies may be required for each situation. If partial scores are taken into consideration, there are 196 basic strategical situations, but only 16 if all partial scores are lumped together. Even these are too many for the "complete and unabridged" texts, but they serve to make the game richer for the critical player. (Those who think the game is already too rich are free to adopt a variety of dodges: (1) re-deal if any hand has no face card, (2) do not play one bids, (3) outlaw doubling, (4) keep the bidding open until the fourth consecutive pass.)

Ignoring partial scores toward game, there are four distinct strategies. This is recognized in "party bridge" where four hands are played at each table, one for each basic situation. But the mathematics is *not* the same as in the natural continuity. Thus, if I am the vulnerable dealer on the second hand and fail to finish the rubber, in natural bridge I shall still be vulnerable on the next deal, but not so in party bridge. Here, to make game when vulnerable is worth 150 more points, when not vulnerable, 50 less points, which may mean a 200 point difference in the appropriate strategy. If you are not the dealer and your hand is a bust, try opening with a two bid and passing the next round! If you are not shot by

Culbertson fans, you will net 400 points per try.

If a two-game rubber scores 700 points while a three-game rubber scores 500 points, what is the unwritten value of the first leg? The mathematical expectation depends on the probability for winning the rubber. If both sides are equally likely to win the second game, the first leg is worth 350 points. For the rubber can be concluded in any of three ways: (1) winning the next game, (2) losing the second game but winning the third, or (3) losing the rubber. For these three events the respective probabilities are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$; the prizes are $+700$, $+500$, -500; and the expectations are the products $+350$, $+125$, -125; so that the net expectation is $+350$.

An easier way to get this answer is to observe that winning the next game establishes a 700 point advantage, while losing gives both sides an equal chance to win the rubber, with no advantage; the average of 700 and zero is 350. The point value of tricks taken, and for honors, is not included in this figure, which represents only the intangible and unrecorded value of winning the first game in a rubber. Similarly, winning the second game of a three-game rubber is also worth 350 points, for it reduces to zero the advantage the other side had. The final game of two-game and three-game rubbers is worth 350 and 500 points respectively, for the recorded scores are 700 and 500, but in the first instance the relative advantage was already 350.

The foregoing computation assumes that the vulnerable and not vulnerable sides are equally likely to win the next game. Because of the higher penalties for overbidding when vulnerable, this assumption is not valid. If the correct probability for the vulnerable side winning is v, a fraction actually less than one-half, the chance to lose is $(1 - v)$, the prizes are 700 and 0, and the mathematical expectation is 700v which is less than 350. (For $v = 40\%$ it would be 280.)

Besides the commendable caution of the vulnerable side, another factor complicates the determination of the expectation, namely the relative skill of the players. Suppose your side has a chance, s, to win the first or third game. Then because of your skill, s, your chance to win the game when only you are vulnerable is no longer v but v^*, where we may choose $v^* = v + 2v(1-v)(2s-1)$. If the situation is reversed your chance to pull even is $(1 + v^* - 2v)$.

Instead of writing down further algebraic formulas, let us use the hypothetical data $s = 75\%$, $v = 40\%$. Then your chance is 48% to win in two games, 36% to win in three games, 12% to lose in three games, and 4% to lose two straight. The more skillful side begins the rubber with a virtual 428 point handicap, but the lead is only 400 if both sides become vulnerable. It is worth 110 points for the stronger side to win the first game, but 330 points to prevent the weaker side from doing so. To prevent the strong team from clinching a two-game rubber is worth only 162 points, but the fair price is 798 points to prevent the weak team from scoring an upset. Professor H. C. Levinson in his *Science of Chance* states that it is "certainly sound play and is so recognized by the leading authorities" to sacrifice 700 points in order to prevent your opponents from winning a two-game rubber. *This is paying bridge blackmail* by at least 300 points, since the price does not neutralize his advantage. We have seen that the fair price for each game is 350 points, plus 150 if both are vulnerable, plus points for tricks and honors. The only mathematical justification for paying more is superior skill (with s greater than one-half), or cessation of play with no credit for the first leg.

Let us conclude with a few more comments pertinent to bidding. Sometimes one partnership does all the bidding, but more frequently both sides compete for the contract. If one game bid of four hearts is overcalled by an opposing bid of five diamonds, it is to be noted that the combined tricks in the auction add up to 21, eight more than the thirteen available. Both bids may be sound, however, for the whole is less than the sum of its parts. Why is this? In bridge, more so than in whist, two long suits form the most aggressive combinations. More frequently one side has strength in three suits. The aggressive purpose of bidding is twofold: to choose the trump suit, and to decide how high to bid; defensively it suggests leads by giving your partner some clue as to the nature of your hand.

All manner of signals have been suggested for the bidding. Systems may come and conventions may go, but here is one principle which is invariant: Each bid selected gives information by virtue of the bidding opportunities which are rejected. Thus (1) an opening bid of "two spades" tells, in any system, that the hand has characteristics which are not so adequately described by "one spade."

Again, (2) bidding first diamonds and then hearts is chiefly significant in that you did not bid hearts first and then diamonds. Since it is a truism to make one's best bid first, and since game at diamonds would require one more trick than game at hearts, the *inverse* bidder obviously deems the diamond suit preferable.

(3) You bid "one no trump" and your partner says "two no trump." He did not pass and settle for a partial score, so why then did he not bid game at "three no trump"? Undoubtedly he is not sure that is the best contract. Since a major trump suit (spades or hearts) is usually a safer game bid than the no trump at one trick cheaper, he either prefers a major or is inviting slam. You say "three no trump" (best, from your viewpoint) and he continues with "four diamonds." Now you know he wants slam!

Or (4) a person starts the bidding after three passes with "one club." Clearly he expects a positive score for a pass would have assured a zero score. If the other side is vulnerable, he undoubtedly is not afraid of hearts or spades, for it would be foolhardy to give them another chance at rubber. No need to multiply examples. To a critical partner each bid is eloquent because of what it does *not* say.

In a pamphlet on "Slam Bidding" Bruelheide, a Culbertson associate, says that slams appear in about 5% of the deals, or about every third rubber, but effect 40% of the score. This added incentive for high bidding furnishes a golden opportunity to sell advice, and systems of "legalized cheating" have mushroomed, particularly since the advent of Contract. A perennial favorite is the one club bid used to signal a weak no trump. Charles H. Goren, in *Point Count Bidding,* argues that this one club bid which does *not* mean clubs is "convenient" and "mandatory," but he denies that this is a "system or convention." Consider this chatter between partners: "Four clubs"; "Four hearts"; "Four spades"; "Five diamonds." In the Gerber Convention this has the code meaning: "How many aces have you?" "Just one!" "How many kings?" "Two."

It is puzzling to witness the feudal allegiance of matinee fans to the "impeccable authority" of the system-makers. One player makes a fourth hand pass with game in hand and apologizes, "I had no bid." Another is proud of the privilege of martyrdom, "I know that a trump lead would have saved a thousand points, but don't you think leading 'fourth highest from your longest and strongest' is best *on the average?"* Open with "three no trump" and three persons at your table argue that there is no such bid; make good this game bid, and you are reprimanded, "We play for the fun of it." Amen! There is fun to be had in exploring the versatile riches of this best of card games. There is fun in bidding and making a grand slam. There is fun also in watching an opponent make good a contract by surprising you out of the tricks you had been so sure of when you doubled. To some there is fun in the security of a popular system. For it is wise, in bridge, as in chess and checkers, when in Nome, do as the Nomads do.